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## *Note de M. Jean Plana sur les pages 60 et 61 du 1^er volume de sa Théorie du Mouvement de la Lune.*

This memoir of M. Plana contains an admission and correction of an error in his Lunar Theory, relative to the secular acceleration of the moon's mean motion, of which it is impossible to over-estimate the importance. Mr. Adams had drawn attention both to the inadequacy of La Place's theory respecting this secular inequality, and to the serious correction which must be applied to the numerical coefficient of $m^4$ in its expression, in a remarkable paper, printed in the *Philosophical Transactions* for 1853, Part III., of which a good extract will be found in the *Proceedings of the Royal Society* for that year. His attention had been drawn to this subject while endeavouring to supply an omission in the theory of the moon given by Pontécoulant in his *Théorie Analytique.* In this work, though the author follows Sir J. Lubbock's method of expressing the co-ordinates of the moon directly in terms of the time, yet he unfortunately adopts Plana's results without examination. Mr. Adams, in performing the calculations requisite to complete the defective part of the theory, was surprised to find that the second term of the secular acceleration thus obtained not only differed considerably in magnitude from the corresponding term given by Plana, but was even of a different sign. Referring then to the *Proceedings of the Royal Society*, page 321, for a sketch of Mr. Adams' processes, it will be sufficient here to give his result, compared with that of Plana. This result is only carried to the second term of the function expressing the acceleration, that is, to the fourth power of $m$ (the ratio of the mean motions of the sun and moon), since it was the author's object to exhibit in its simplest form the error which he had discovered. but he at the same time expressed his intention of carrying his approximation still farther, so as to obtain the coefficients with an accuracy sufficient for the calculation of remote eclipses.

The expression obtained by Mr. Adams for the first two terms of the acceleration, was:—

$$-\left(\frac{3}{2}m^2 - \frac{3771}{64}m^4\right) \int (e'^2 - E'^2)\, n\, d\, t$$

while, according to Plana's *Théorie de la Lune*, the corresponding expression is,

$$-\left(\frac{3}{2}m^2 - \frac{2187}{128}m^4\right) \int (e'^2 - E'^2)\, n\, d\, t$$

The difference of the two expressions is,

$$\frac{5355}{128}m^4 \int (e'^2 - E'^2)\, n\, d\, t$$

and the numerical value of the correction is,

$$-\ 1''\cdot 66 \left(\frac{t}{100}\right)^2$$

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or — 1″·66 multiplied by the square of the number of centuries from a fixed epoch.

In the year 1854, M. Plana published in the *Turin Memoirs* a short note in which he discusses the result arrived at by Mr. Adams. After indicating the pages in his *Théorie*, in which will be found the details of the calculations for the completion of the second term of the function which expresses the secular acceleration, he remarks, "I cannot establish a complete comparison between the result of Mr. Adams and my own, without again going through all the details; which would require a very painful labour, from which I think I may hold myself excused, by the care with which I have published all the intermediate results. By means of them we can always keep before the eyes the proof of the definitive results, upon which we ordinarily concentrate all the importance which they deserve, on the supposition that they are incontestable, whatever may be the complication inherent in their deduction." After a few words of explanation respecting his method, tending to prove that no essential term depending upon the secular variations of the orbit of the earth has been omitted, he concludes his note by recommending the part of his own theory under discussion "to the judgment of all those who would be willing to submit it to a profound and severe examination."

It appears, therefore, that at the time of writing this note, M. Plana had no suspicion of anything defective either in his own processes or in the theory of La Place, nor has any astronomer, from that time to the present, followed his recommendation "to study profoundly the complicated processes" on which his result for the secular acceleration is based. Fortunately, however, the author has recently found occasion to do this himself, and has not only discovered the source of the error, which occasions the discrepancy between his own expression and that of Mr. Adams, but with most commendable activity and energy, has thoroughly traced the effect of it in every term where it produces any sensible effect, and gives the result of his researches in a memoir, which is the chief subject of this article. The chief point for observation is that the correctness of Mr. Adams' expression for the secular acceleration to the second term of the function is fully verified, and thus the necessity is recognised of applying a correction to the mean motion of the moon, which will interfere materially with all recent chronological speculations, and make a recomputation of the elements of all ancient eclipses imperative.

As usual, the source of the error was very simple and obvious, though, singularly enough, it has escaped the notice of all astronomers. M. Plana's account of it is as follows. In treating of the secular variations of the elements of the solar orbit, he has occasion to discuss the integral expressed by the equation,

$$x = \mathrm{B} \int \varepsilon'^{m} \sin (k v + \beta - n \tau') \, d v$$

which, he says, comprises all the cases of integration which can

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present themselves in the research of the co-ordinates of the moon. By means of processes, which it is not necessary to give in detail, he finally arrives at the following equation expressing the value of $x$,—

$$x = -\frac{\mathrm{B}}{k}\,\varepsilon'^{m}\cos(k\nu+\beta-n\tau') - \frac{\mathrm{B}\,n\,d\tau'}{k^{2}\,d\nu}\,\varepsilon'^{m}\cos(k\nu+\beta-n\tau') + \frac{\mathrm{B}\,m}{k^{2}}\,\varepsilon'^{m-1}\,\frac{d\varepsilon'}{d\nu}\sin(k\nu+\beta-n\tau')$$

but by a strange typographical error the last of the three terms given above was omitted in the text near the end of page 61 of the 1st volume of the *Théorie de la Lune.* That this error was purely typographical is evident from the words which follow, namely, "l'on peut réunir ces trois termes en un seul, et poser

$$x = -\frac{\mathrm{B}\,e'^{m}}{k-n\dfrac{d\tau'}{d\nu}}\cos\left\{k\nu+\beta-n\tau'+\left(k-n\frac{d\tau'}{d\nu}\right)\frac{m}{\varepsilon'\,k^{2}}\cdot\frac{d\varepsilon'}{d\nu}\right\}$$

" Thus," he says, "it happened, that, in composing the note of the 17th December, 1854, I did not pay attention to the omission of the term

$$\frac{\mathrm{B}\,m}{k^{2}}\,\varepsilon'^{m-1}\,\frac{d\varepsilon'}{d\nu}\sin(k\nu+\beta-n\tau')$$

which should form part of the value of $x$ which immediately precedes the last. In consequence of this omission I saw only the term multiplied by

$$\varepsilon'^{m}\,\frac{d\tau'}{d\nu}$$

and I considered that the one multiplied by

$$m\,\varepsilon'^{m-1}\,\frac{d\varepsilon'}{d\nu}$$

was destroyed by others. To this circumstance must be attributed my method of recognising (*ma manière de voir*) the non-existence of the periodical terms multiplied by $\frac{d\varepsilon'}{d\nu}$ in the expressions of $\delta u$ and $\delta n t$. As soon as I had re-established, by performing again the calculations of page 61, the true trinomial in the place of the monomial equal to the value of $x$, I saw that the integration will introduce secular terms into the integral, namely, products of the form

$$x\,\mathrm{B}'\,\varepsilon'^{m}\sin(k\nu+\beta-n\tau')\,d\nu:$$

for there will be the term

$$\int\frac{\mathrm{B}\,\mathrm{B}'m}{k^{2}}\,\varepsilon'^{2m-1}\,\frac{d\varepsilon'}{d\nu}\sin^{2}(k\nu+\beta-n\tau')\,d\nu$$

$$=\int\frac{\mathrm{B}\,\mathrm{B}'m}{2\,k^{2}}\,\varepsilon'^{2m-1}\,d\varepsilon' = \frac{\mathrm{B}\,\mathrm{B}^{1}}{4\,k^{2}}\,\varepsilon'^{2m}.$$

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Hence the existence of terms of the form

$$G\,\varepsilon'^{\,2m-1}\,\frac{d\,\varepsilon'}{d\,\nu}\,(\sin k\,\nu + \beta - n\,\tau')$$

in the expression of $\delta u$ is incontestable. Their form is similar to that of the term $Q \sin(i\,\nu + \beta)$ of which La Place has remarked the existence at page 214 of the third volume of the *Mécanique Céleste.*"

M. Plana then proceeds to investigate the secular terms which will arise from the omitted term of his trinomial; and, after an investigation of considerable length, arrives at precisely the same result as Mr. Adams had obtained by a different process.

He then proceeds with the investigation of three other terms of the same order, arising from the omitted term of the trinomial.

The first of these, which is to replace the corresponding term in the secular equation of the mean motion at page 485 of the first volume of the *Théorie de la Lune,* is

$$-\frac{99}{128}\,m^2 e^2 \int (\varepsilon'^2 - E'^2)\,d\nu$$

the term, as it stands at the place cited, being

$$+\frac{1461}{128}\,m^2 e^2 \int (\varepsilon'^2 - E'^2)\,d\nu$$

The second term, which must replace the corresponding one on the same page as before, is

$$+\frac{447}{128}\,m^2 \gamma^2 \int (\varepsilon'^2 - E'^2)\,d\nu$$

instead of

$$+\frac{525}{128}\,m^2 \gamma^2 \int (\varepsilon'^2 - E'^2)\,d\nu$$

Finally, the third term, which must replace the corresponding one in the *Théorie* (same page as before), is

$$-\frac{75}{32}\,b^4 \int (\varepsilon'^2 - E'^2)\,d\nu$$

instead of

$$+\frac{75}{32}\,b^4 \int (\varepsilon'^2 - E'^2)\,d\nu$$

Hence the correct expression for the secular equation of the mean motion of the moon to terms of the fourth order, is as follows:—

$$\int \zeta\,d\nu = \left(\frac{3}{2}m^2 - \frac{3771}{64}\,m^4 - \frac{99}{128}\,m^2 e^2 + \frac{447}{128}\,m^2 \gamma^2 - \frac{75}{32}\,b^4\right) \int (\varepsilon'^2 - E'^2)\,d\nu$$

$$+\frac{15}{8}\,m^2 \int (\varepsilon'^4 - E'^4)\,d\nu \quad *$$

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* In Plana's Memoir, this term has for its coefficient $\frac{15}{2}$, which is presumed to be error.

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instead of

$$\int \zeta \, d\nu = \left(\frac{3}{2}m^2 - \frac{2187}{128}m^4 + \frac{1461}{128}m^2 e^2 + \frac{525}{128}m^2 \gamma^2 + \frac{75}{32}b^4\right)\int (\varepsilon'^2 - E'^2)\, d\nu$$
$$+ \frac{15}{8}m^2 \int (\varepsilon'^4 - E'^4)\, d\nu$$

as it stands in the *Théorie*, vol. ii. p. 852.

With regard to these results M. Plana observes :—

" The calculation of the last three terms requires considerations more delicate than those for the term multiplied by $m^4$. The agreement which I now obtain with respect to the term multiplied by $m^4$, proves the correctness of my preceding calculations; otherwise it would have been impossible to cause the discordance to disappear by this *simple addition* of the terms, which I have calculated in this note, by the assistance of my formula [1] resulting from the analysis which I have explained in pages 60 and 61 of the first volume of my work."

M. Plana, after obtaining by analysis the periodical terms multiplied by $\frac{d\varepsilon'}{d\nu}$ in the expression for the mean longitude, and showing that they are insensible, concludes with the following remarks :—

" I reserve for another time the calculation of the terms of the fifth, sixth, and seventh orders, which ought to be added to those which I have already calculated, in order to complete, to the last order at least, the co-efficient of the secular equation of the mean motion of the moon. The formulæ which I have established in this note are important for directing the choice of arguments and of the terms of their co-efficients, which must be considered in order to comprise in this definitive result all the quantities of the same order. For the present, it is sufficient for me to have completed the calculation of terms of the fourth order, indicating at the same time the sources from which may be derived the following terms by the aid of the different functions which are developed in my work. This greater precision is necessary that we may be in a condition to verify, not only by observations of eclipses, but also by those of the *libration* separated by intervals of several thousand years, that the secular equation in question is communicated by the attraction of the earth to the mean motion of rotation of the lunar spheroid with such uniformity, that it will always be impossible to see the hemisphere opposite to that which the moon now presents to us."